Solving Kepler's Equation and Calculating the Positions of the Planets (An Overview)

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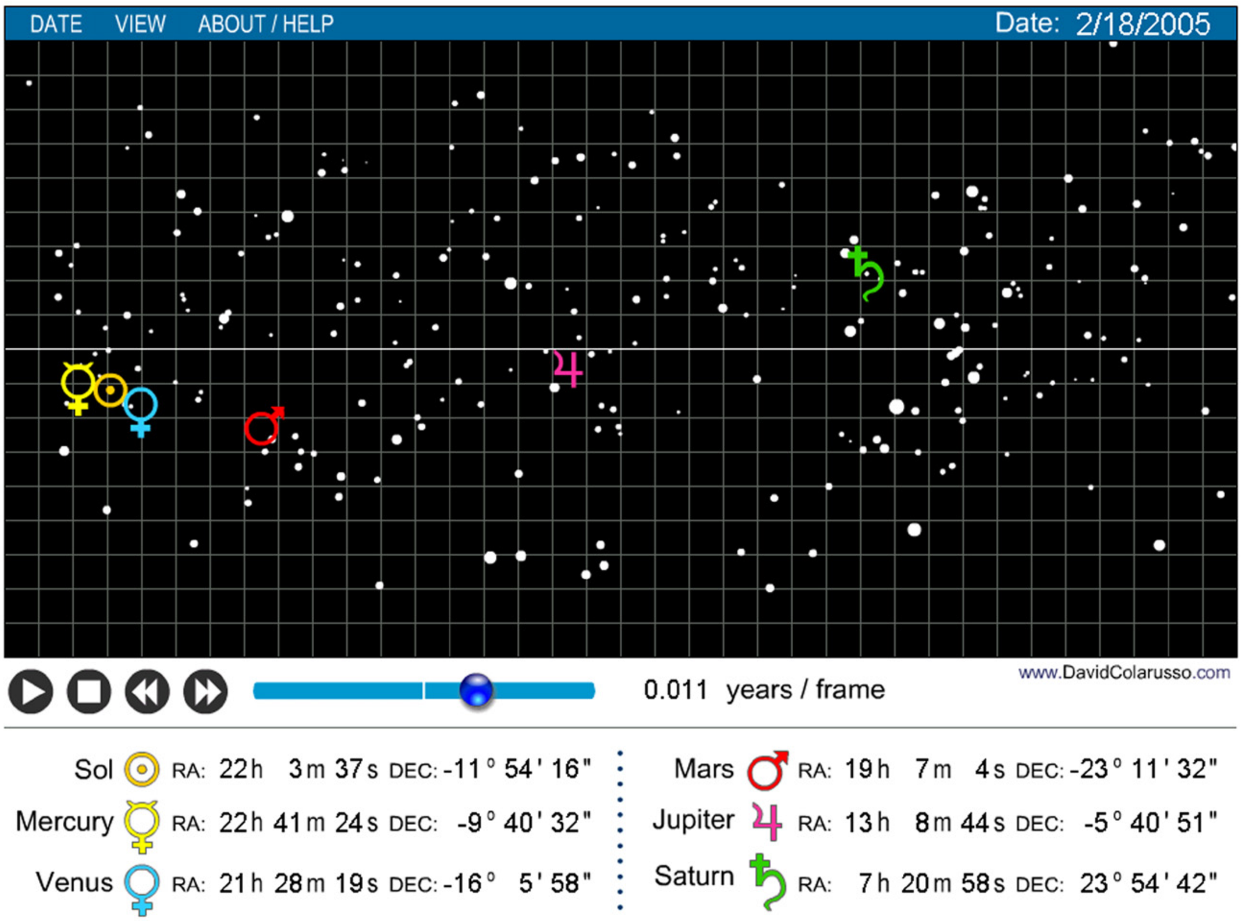

This brief explanation describes the methods used in the simulation found at www.DavidColarusso.com to determine the **Right Ascension** (RA) and **Declination** (Dec) of the Sun, Mercury, Venus, Mars, Jupiter, and Saturn. It assumes some background in astronomy. However, there is a reasonably verbose glossary at the back for what may be unfamiliar terms. These terms will appear in **bold**. In truth, this explanation is more for those wishing to know exactly what assumptions underlie the above simulation. Teachers and interested browsers will find a set of educational material directed at a broader audience under the *Build Your Own Solar System* part if the site. However, this explanation does serve as a good case study in how to determine the planets' positions.

The simulation overlays the RA and Dec of the five planets visible to the naked eye and the sun atop an **equal area projection** of the Earth's sky. The name planets is Greek for "wanderers." Knowing the **geocentric** model held for so long, many of my students simply assume the planets move about the sky in much the same manner as the Moon or Sun–steadily creeping eastward. They do not, and I developed this simulation to illustrate their true wandering nature.

1 Solving Kepler's Equation and Calculating Ephemerides

When first approaching this problem, I knew that I would need to solve **Kepler's Equation** and have a little fun with reference frames. I was surprised by the explanations available on the web. Most of them fell short of what I was looking for. They were either strictly qualitative or, if quantitative, unnecessarily opaque. This resulted mostly from having too few or poorly executed diagrams and illustrations. There were a few notable exceptions, and I have cited them in the references section. As I never formally studied celestial mechanics, these sources were my teachers and greatly helped acquaint me with the problem and its solutions. What I've attempted to do here is to weave together a relatively cogent "How to" on the solving of Kepler's Equation and the calculation of planetary positions in the Earth's sky.

Johannes Kepler (1571-1630) was a mathematician, astronomer, and **Copernican**. He believed that the Sun, not the Earth, lay at the center of the universe. He refined Copernicus's view of a **heliocentric** (Sun-centered) universe, making it into more than simply a competing theory for the **geocentric** (Earth-centered) model. Under Kepler it would become the superior predictive model. In his work Kepler formulated three laws of planetary motion first set down together in Harmonice Mundi (Harmonies of the World), 1618, and here they are.

Kepler's Laws

1. The planets orbit about the Sun in **elliptical** orbits with the sun centered at one of the **ellipse**'s two **foci** (Figure 1).
2. An imaginary line connecting the sun and a planet sweeps out equal areas in equal times as the planet moves through its orbit. A consequence of this is that a planet moves fastest when closest to the Sun. *Newton will have something to say about this.*
3. The square of the **period** of a planet's orbit is proportional to its distance from the Sun cubed. When the units used for distance are **Astronomical Units** (AU) and time is measured in years, this relationship can be written explicitly as an equation relating the planet's period P and the **semi-major axis** of its orbit a (eq.1).

$$P^2 = a^3 \tag{1}$$

Kepler's Laws meant that given only a handful of **orbital parameters**, one could say where a planet had been and would be. To state this explicitly, astronomers make use of Kepler's Equation (eq.2).

$$M = E - esinE \tag{2}$$

M = mean anomaly
E = eccentric anomaly
e = eccentricity of orbit

Kepler's equation is a **transcendental equation**. This means there is no general solution. So to find the location of a planet at a time t, we must solve for that time using some **numerical method**. First let us work with what we have. *NOTE: You may find it helpful to reference Figure 2 to help visualize some of the variables referenced here.*

Only e is time independent. So we consult our orbital parameters for its value and then solve for the **mean anomaly** (eq.3), M in Kepler's Equation (eq.2). The mean anomaly is just the angle with the **perihelion** that the planet would have if the orbit was an ellipse with **eccentricity** = 0, i.e., a circle. We call the

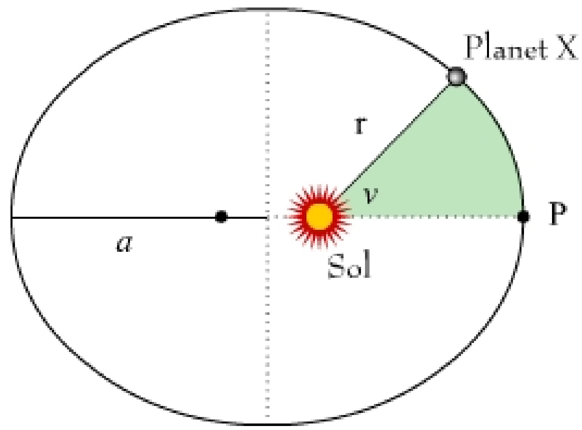

Figure 1: Planet X on its elliptical orbit about the Sun (Sol).

a = semi-major axis of the planet's orbit
r = distance from Sun to planet
P = perihelion (point of closest passage to the Sun)
ν = angle of planet X from perihelion

imaginary planet moving along such an orbit the **mean planet**. In such a case the planet would move with a **velocity** $V = \frac{(2\pi)}{period}$.

$$M = \frac{2\pi(t-T)}{P} \tag{3}$$

P = period of orbit (i.e., $P = \sqrt{a^3}$)
t = time for which you are solving
T = time of last perihelion passages

As you can see, the mean anomaly is just the mean planet's velocity times the time elapsed since it was last at the perihelion.

We can now find the **eccentric anomaly** using some **numerical method**. This simulation makes use of **successive approximation**. Once we have a value for E with which we are happy, we can find the **true anomaly** (eq.4). The true anomaly is the ACTUAL angle between the perihelion and the planet.

$$\nu = 2\arcsin\left(\sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}\right) \tag{4}$$

From here it is a simple matter to find the planet's **radial distance** (eq.5) from the Sun.

$$r = \frac{a(1-e^2)}{1+e\cos\nu} \tag{5}$$

We now have the planet's **polar coordinates** (r, v) within the plane of its orbit such that the X axis points from the Sun towards the Perihelion, point P.

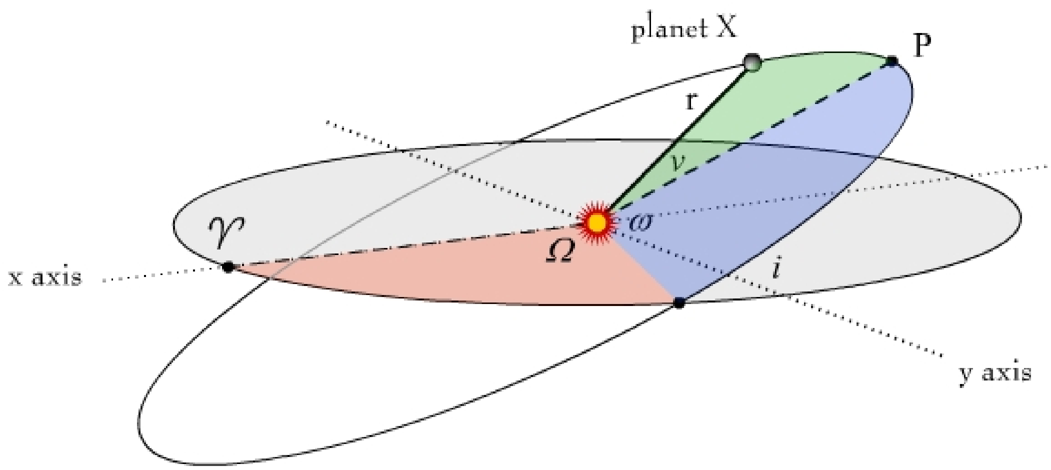

Figure 2: Planet X's orbit as it intersects with the plane of the Sun's equator.

r = distance between planet X and the Sun
P = perihelion (point where X is closest to the Sun)
Υ = first point of Aries
i = angle between plane of Sun's equator and the planet X's orbit
ω = longitude of perihelion
Ω = longitude of asending node

Now we find the **Heliocentric Ecliptic coordinates** (x, y, z) for the planet by converting from polar to cartesian coordinates and rotating the frame such that the X axis points towards the **first point of Aries**.

$$x = r(\cos\Omega\cos(\omega+\nu) - \sin\Omega\sin(\omega+\nu)\cos i) \tag{6}$$
$$y = r(\sin\Omega\cos(\omega+\nu) + \cos\Omega\sin(\omega+\nu)\cos i) \tag{7}$$
$$z = r\sin(\omega+\nu)\sin(i) \tag{8}$$

We then rotate the coordinates into **Heliocentric Equatorial coordinates** (X, Y, Z), making use of a rotational matrix.

$$X = x \tag{9}$$
$$Y = y\cos(i) - z\sin(i) \tag{10}$$
$$Z = y\sin(i) + z\cos(i) \tag{11}$$

However, our display shows the positions of the planets from the Earth. So we need to switch our vantage point to that of a geocentric system. To do this we first repeat the above process, solving for the Earth's Heliocentric Equatorial coordinates. We want to know the Sun's Geocentric coordinates. So here we will approximate this as the inverse of Earth's heliocentric coordinates. This is the same method used to find the Sun's location for our display. It is important to note, however, that this is just an approximation, as what we really find is not the location of the Earth but rather that of the Earth-Moon system's **barycenter**. This simplification is responsible for limiting the simulation's accuracy. *Note: This is not an issue for* *the Build Your Own Solar System simulation for teachers as the hypothetical "Earth" has no moon in that simulation.* We then add the Sun's geocentric coordinates to those of the heliocentric coordinates of our planet. This shifts the coordinates, giving us the **Geocentric equatorial coordinates** (x_p, y_p, z_p) for the planet.

$$x_p = X + x_\odot \tag{12}$$
$$y_p = Y + y_\odot \tag{13}$$
$$z_p = Z + z_\odot \tag{14}$$

Having the planet's Geocentric coordinates, it is a simple matter to convert them into Right Ascension and Declination. *Note: Watch you signs here; if you are not careful, it WILL get messy.*

$$RA = \arctan \frac{y_p}{x_p} \tag{15}$$
$$Dec = \arctan \frac{z_p}{\sqrt{x^2 + y^2}} \tag{16}$$

That's it. We can now solve for many discreet times and collect the data into tables to construct **ephemerides**. If you are interested in finding more accurate calculations for the planets' positions, consider buying a copy of the *Astronomical Almanac* from the US Naval Observatory or making use of JPL's Horizons system (see references).

References:

1. To anyone interested in why it is the orbits of the planets are elliptical, I suggest finding a copy of D. & J. Goodstein's *Feynman's Lost Lecture: The Motion of Planets Around the Sun*. W. W. Norton & Company. New York, NY. 1996.

2. A copy of Kepler's *Harmonice Mundi (Harmonies of the World)* as well as many other ground breaking texts in astronomy have been compiled into one tome: Stephen Hawking's *On The Shoulders of Giants: The Great Works of Physics and Astronomy*. Running Press. Philadelphia. 2002.

3. For what I found to be the most rigorous on-line handling of this material, try Dr. J. B. Tatum's Celestial Mechanics: http//orca.phys.uvic.ca/ tatum/celmechs.html (Link current as of April 2004).

4. The orbital parameters used here came from the JPL Solar System Dynamics Group's "Planetary Orbital Elements," *JPL Solar System Dynamics*: http//ssd.jpl.nasa.gov/elem_planets.html. (Link current as of April 2004).

Glossary of Terms

Ascending node: The point of intersection between a planet's orbit and the plane of the Sun's equator, where the planet is moving northward ("up") across the plane of the Sun's equator.

Astronomical Units (AU): A measure of distance where one AU = the semi-major axis of the Earth's orbit around the sun. 1 AU = —,000,000 miles.

Barycenter: The center of mass for a multi-body system of mutually orbiting bodies. The system orbits about the barycenter.

Copernican: One who subscribes to the Copernican world view of a heliocentric universe, i.e., one who believes that the Earth orbits around a fixed Sun.

Declination (DEC): A heavenly object's position in the sky as measured along a meridian in degrees (0 to 90 degrees) north (+) or south (-) from the equator.

Descending node: The point of intersection between a planet's orbit and the plane of the Sun's equator, where the planet is moving southward ("down") across the plane of the Sun's equator.

Eccentric Anomaly:

Eccentricity: A measure of how "elliptical" an eclipse is. For example, a circle has an eccentricity of zero, not very elliptical. A relationship can be stated mathematically between the semi-major axis a, the semi-minor axes b and the eccentricity e where $b^2 = a^1(1 - e)$

Ephemerides: plural of ephemeris. Tables containing the positions (usually RA and DEC) of celestial objects for different times, usually at regular intervals.

Ellipse: One of the conic sections, those shapes which are the intersection of a cone and plane, the ellipse is a geometric shape that looks like a squashed circle. You can easily make an ellipse with two thumb tacks and a loop of string. Place the two tacks into a paper and loop the string around them. Place a pencils in the loop of string and move it outwards until the loop becomes taught. Move the pencil around the tacks always keeping the slack out of the loop. The figure drawn is an ellipse. The points where the thumbtacks lie are the foci of the ellipse.

Elliptical:

Equal area projection:

First point of Aries: The position against the background stars of the Earth's descending node (the vernal equinox) as seen from the Sun.

Foci: Plural of focus. See ellipse.

Geocentric: Earth centered.

Geocentric Equatorial Coordinates: An X,Y,Z coordinate system centered on the Earth in which the Earth's equator lies in the X-Y plane.

Heliocentric: Sun centered.

Heliocentric Ecliptic Cartesian coordinates: An X,Y,Z coordinate system centered on the sun in which the ecliptic lies in the X-Y plane.

Heliocentric Equatorial Cartesian coordinates: An X,Y,Z coordinate system centered on the sun in which the sun's equator lies in the X-Y plane.

Kepler's Equation: An equation discovered by Johannes Kepler (1571-1630) whose solution can specify the position of a planet in its orbit for a specified time.

Matrix:

Mean anomaly: The angle between the perihelion and the mean planet as measured in the plane of its orbit.

Mean planet: An imaginary planet which moves at a constant velocity around a circular orbit with a radius equal to the semi-major axis of the actual planet's orbit.

Numerical method: A method for solving transcendental equations which guesses a value for a variable plugs it in to the equation and sees if it works. . . . .

Orbital parameters: A set of physical parameters for the orbit of a planet sufficient to predict the position of the planet at a given time t.

Perihelion: The closest point on a planet's orbit to the Sun.

Period (of a planet): The length of time it takes a planet to return to the same place in its orbit.

PI: Also denoted by the Greek letter π. The ratio of a circle's circumference C to its diameter D. $\pi = \frac{C}{D}$

Polar Coordinates: A means of denoting a point's location by use of its radial distance from the origin and the angle it is from the x axis.

Radial distance: How far something is from a coordinate axes as measured directly out from the zero point.

Right Ascension (RA): A heavenly object's position in the sky as measured in hours:minutes:seconds east (+) or west (-) from the first point of Aries.

Semi-major axis:

Semi-minor axis:

Successive approximation: A numerical method by which a solution is fond to an equation by substituting in guesses for the answer on both sides of the equation. The sides are evaluated and the first guess that produces a difference between sides of less than a pre-defined tolerance is take to be the answer.

Sol: our sun.

Transcendental equation: An equation for which a general solution can not be found.

True Anomaly: The angle between the perihelion and the planet as measured in the plane of its orbit.

Vector:

Velocity: A measure of an object's motion that includes both the object's speed and direction.

Vernal equinox: